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On the scattering of sound by a magnetic field in a MHD fluid

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Abstract. We investigate the effect of a localized magnetic field on the propagation of sound in an infinite fluid described by the magnetohydrodynamic equations (MHD). An externally imposed magnetic field will scatter an acoustic wave, and the scattered wave is related to the spatial structure of the magnetic field. Measuring it is thus a non-intrusive probe for the magnetic field. Simple examples likely to be encountered in practice are worked out, and estimates are given that suggest the practical feasability of this diagnostic tool in current MHD experiments.

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The propagation of a wave in a non homogeneous medium generates a scattered wave which carries information about the structure of the objects responsible for the scattering. This basic fact has been widely used in condensed matter physics to study the distribution of atoms through X-ray scattering, and the distribution of magnetic moments through neutron scattering.

More recently, interaction of acoustic waves with hydrodynamics flows has been investigated. The velocity gradients in a flow scatter acoustic waves and the scattering pattern is related to the flow vorticity. Based on this concept non-intrusive probes of the vorticity have been constructed in order to study swirling and turbulent flows [1].

Electrically conducting fluids described by the MHD equations are quite widespread and a particulary interesting example is a liquid metal. The flow of such a liquid in the Earth interior is responsible for magnetic field generation by dynamo effect [2]. In the last years, many attempts to see this effect in laboratories have been made [3]. In these facilities, the possibility of using a non-intrusive probe for the magnetic field is quite interesting and following sound-flows interaction ideas, it is natural to investigate the effect of a magnetic field on the sound propagation. In the following, the discussions of orders of magnitude are restricted to the case of liquid metals but notice that the calculations are valid for any fluid described by the MHD equations, in particular for plasmas described in the MHD framework.

In this note we report results of an investigation into the interaction of acoustic waves with magnetic field in a fluid described by the MHD equations. This problem has been studied by Lyamshev and Skvortsov [4], who showed that the magnetic field will scatter the acoustic wave, and that the scattering amplitude is proportional to the Fourier transform of a quadratic function of the magnetic field. Their formulation is valid in three dimensions and for a static field. In this paper we provide an alternative derivation of Lyamshev and Skvortsov's result [4], and extend it to the case of a time dependent magnetic field and for two dimensional situations. In the acoustic limit, where inertia terms can be neglected in the Navier-Stokes equation, we derive the wave equation obeyed by the sound if a magnetic field is applied. This equation for the acoustic velocity field is written in an equivalent, integral equation form, using the Green function of the free-propagating wave equation. In the first order of the Born approximation and in the far-field limit [1], the scattered wave is calculated and is a function of the imposed magnetic field. General properties of the scattering are presented and some patterns are discussed for simple 2-D configurations.

The velocity field $\mathbf{v}$, the magnetic field $\mathbf{B}$ and the pressure field P in an electrically conducting fluid with density ρ, electrical conductivity σ, kinematic viscosity ν follow the equations

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t}+\mathbf{v}\cdot\nabla\mathbf{v}\right)=-\nabla P+\rho\nu\nabla^2\mathbf{v}+\frac{(\nabla\times\mathbf{B})\times\mathbf{B}}{\mu_0} \quad (1)$$

$$\frac{\partial \mathbf{B}}{\partial t}=\nabla\times(\mathbf{v}\times\mathbf{B})+\frac{1}{\mu_0\sigma}\nabla^2\mathbf{B},\quad \nabla\cdot\mathbf{B}=0, \quad (2)$$

$$\frac{\partial \rho}{\partial t}+\nabla\cdot\rho\mathbf{v}=0, \quad (3)$$

where μ_0 is the magnetic permeability of the vacuum. This set of equations is usually completed with a thermodynamic equation that relates the pressure and the density $P(\rho)$. Assume that the fluid is submitted to an external magnetic field $\mathbf{B}_0$. The fluid density at rest is ρ_0 and the isentropic compressibility is χ. We suppose that the magnetic field does not create any flow so the basic state of the fluid is at rest. This is the case when the Laplace force is exactly balanced by the pressure force. We write $\mathbf{B} = \mathbf{B}_0 + \mathbf{B}_s$, $P = P_0 + P_s$, $\rho = \rho_0 + \rho_s$, $\mathbf{v} = \mathbf{v}_s$. The terms X_0 are related to the fluid at rest whereas X_s stands for the medium perturbation due to the acoustic wave, and thus are considered to be small compared to the former ones. The magnetic diffusivity of the fluid is $\eta = (\mu_0\sigma)^{-1}$ and $c = (\chi\rho_0)^{-1}$ is the sound velocity in the fluid. We write the equations at first order in the perturbation fields

$$\rho_0 \frac{\partial \mathbf{v}_s}{\partial t} = -\nabla P_s + \rho_0 \nu \nabla^2 \mathbf{v}_s + \frac{1}{\mu_0}(\nabla \times \mathbf{B}_0) \times \mathbf{B}_s + \frac{1}{\mu_0}(\nabla \times \mathbf{B}_s) \times \mathbf{B}_0, \tag{4}$$

$$\frac{\partial \mathbf{B}_s}{\partial t} = \nabla \times (\mathbf{v}_s \times \mathbf{B}_0) + \eta \nabla^2 \mathbf{B}_s, \ \ \nabla \cdot \mathbf{B}_s = 0 \tag{5}$$

$$\frac{\partial \rho_s}{\partial t} = -\rho_0 \nabla \cdot \mathbf{v}_s, \ \ P_s = c^2 \rho_s. \tag{6}$$

The last equation is the thermodynamic relation for isentropic evolution. A diffusive term appears in (5). For an acoustic wave with frequency ω, it can be neglected if $\omega \ll c^2/\eta$. For common liquid metal, like liquid sodium, we have $\eta \simeq 0.1\ \mathrm{m^2\,s^{-1}}$ and $c \simeq 10^3\ \mathrm{m\,s^{-1}}$ such that this diffusive effect can be neglected for frequencies smaller than 10^5 Hz. Another diffusive term is related to the viscous dissipation. It can be neglected if $\omega \ll c_a^2/\nu$ where $c_a = B_0(\rho_0\mu_0)^{-1/2}$ is the velocity of the Alfven waves that can propagate in a conducting medium penetrated by a magnetic field of amplitude B_0 [5]. For liquid sodium $\rho_0 \simeq 10^3\ \mathrm{kg\,m^{-3}}$, $\nu \simeq 10^{-6}\ \mathrm{m^2\,s^{-1}}$ and for a magnetic field $B_0 = 1\,T$, viscous dissipation can be neglected for frequencies smaller than 10^8 Hz. This constraint for neglecting viscous effects is less stringent than the former one.

For frequencies such that we can neglect the dissipation terms, taking the time derivative of (4), the gradient of (6), substracting one from the other and using (5) leads to a wave equation for the acoustic velocity field:

$$\frac{1}{c^2}\frac{\partial^2 \mathbf{v}_s}{\partial t^2} - \nabla(\nabla \cdot \mathbf{v}_s) = S(\mathbf{B}_0, \mathbf{v}_s), \tag{7}$$

where S is the sound-magnetic field interaction term,

$$S(\mathbf{B}_0, \mathbf{v}_s) \equiv \frac{1}{\rho_0 \mu_0 c^2}\big(-\nabla\left(\mathbf{B}_0 \cdot \nabla \times (\mathbf{v}_s \times \mathbf{B}_0)\right) + \left(\mathbf{B}_0 \cdot \nabla\right) \nabla \times (\mathbf{v}_s \times \mathbf{B}_0) + \left(\nabla \times (\mathbf{v}_s \times \mathbf{B}_0) \cdot \nabla\right) \mathbf{B}_0 \big).$$

The inhomogeneous wave equation (7) can be turned into an integral equation:

$$\mathbf{v} = \mathbf{v}_i + \mathcal{G} * S(\mathbf{B}_0, \mathbf{v}), \tag{8}$$

where $\mathbf{v}_i$ is solution of the homogeneous problem and the other term is the convolution of the free propagation Green function $\mathcal{G}(\mathbf{r}, t, \mathbf{r}', t')$ with the coupling term S. The Green function is a tensor that determines how a velocity perturbation in $(\mathbf{r}', t')$ propagates at $(\mathbf{r}, t)$. Its coefficients in a Cartesian frame are solutions of

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathcal{G}_{\alpha\beta} - \partial_\alpha \partial_\mu \mathcal{G}_{\mu\beta} = \delta(\mathbf{r} - \mathbf{r}')\delta(t - t')\delta_{\alpha,\beta} \tag{9}$$

where $\delta(z)$ is Dirac's delta function, $\delta_{\alpha,\beta}$ is the Krönecker symbol and a summation is made on repeated indices. The solution of (9) in homogeneous space is only a function of relative position and time $\mathcal{G}_{\alpha\beta}(\mathbf{r}, t, \mathbf{r}', t') = \mathcal{G}_{\alpha\beta}(\mathbf{r} - \mathbf{r}'; t - t')$ and, in three dimensions, it is given by [6]

$$\begin{aligned}\mathcal{G}_{\alpha\beta}(\mathbf{r}; t) = \frac{c^2}{4\pi}&\left(\partial_\alpha \partial_\beta \left(\frac{1}{r}\right)\right. \\ &\left.-\delta_{\alpha\beta}\nabla^2\left(\frac{1}{r}\right)\right)\int_0^t (t - \tau)\delta(\tau)d\tau \\ &-\frac{c^2}{4\pi}\partial_\alpha \partial_\beta \left(\frac{1}{r}\right)\int_0^{r/c} \tau\delta(t - \tau)d\tau \\ &+\frac{1}{4\pi}\frac{r_\alpha r_\beta}{r^3}\delta(t - r/c).\end{aligned} \tag{10}$$

For our purposes we need only the slowest decaying (in space) component of this expression, the so-called far-field term, given by the last term on the right of (10):

$$\mathcal{G}_{\alpha\beta}(\mathbf{r}, t, \mathbf{r}', t') \approx \frac{\delta\left(t - t' - \frac{\|\mathbf{r} - \mathbf{r}'\|}{c}\right)}{4\pi\|\mathbf{r} - \mathbf{r}'\|} u_\alpha u_\beta, \tag{11}$$

where $\mathbf{u} = (\mathbf{r} - \mathbf{r}')/(\|\mathbf{r} - \mathbf{r}'\|)$ and u_α its components. This expression is consistent with the intuition that only components of the perturbation parallel to $\mathbf{u}$ propagate because in a normal fluid, transverse perturbations do not propagate, and at $\mathbf{r}$ the velocity perturbation is longitudinal and thus parallel to $\mathbf{u}$. Consequently, the tensorial part of the far-field Green function is simply a projection on $\mathbf{u}$.

Substitution into equation (7) and taking the limit of distances large compared to the size of the region where the interaction between acoustic wave and magnetic field takes place, $\|\mathbf{r} - \mathbf{r}'\| \gg \|\mathbf{r}'\|$, with $\mathbf{u} \simeq \mathbf{r}/\|\mathbf{r}\|$, we get

$$\begin{aligned}(\mathcal{G} * S(\mathbf{B}_0, \mathbf{v}))(\mathbf{r}, t) &= \frac{1}{8\pi^2 r}\int d\nu e^{i\nu(\|\mathbf{r}\|/c - t)} H\mathbf{u}, \\ H &= \int d\mathbf{r}'\, dt' e^{i\nu(t' - \mathbf{u}\cdot\mathbf{r}'/c)}(S(\mathbf{B}_0, \mathbf{v}) \cdot \mathbf{u})(\mathbf{r}', t').\end{aligned} \tag{12}$$

As in other scattering problems, the perturbation in the medium emits spherical waves whose amplitudes are related to the Fourier transform of the source term in (7). If the source term decreases fast enough away from the region where the magnetic field is applied, $\mathbf{v}_i$ is identified as the incident wave and $\mathcal{G} * S(\mathbf{B}_0, \mathbf{v})$ is the scattered wave. In the Born approximation, this wave is small compared to the incident wave and can be calculated perturbatively

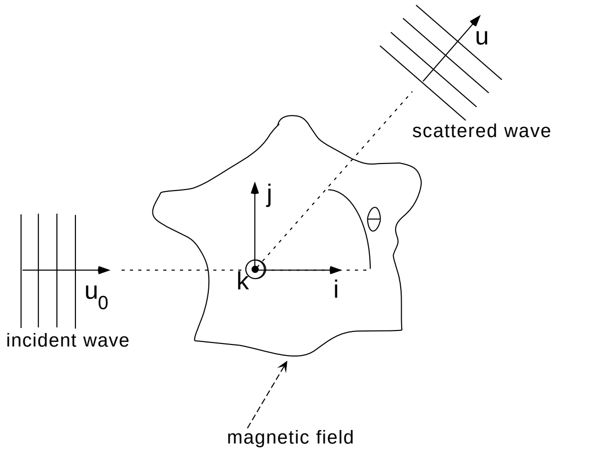

Fig. 1. Sketch of a sound scattering experiment on a localized magnetic field. $\mathbf{u}_0$ is the direction of the incident wave and $\mathbf{u}$ is the direction of scattering.

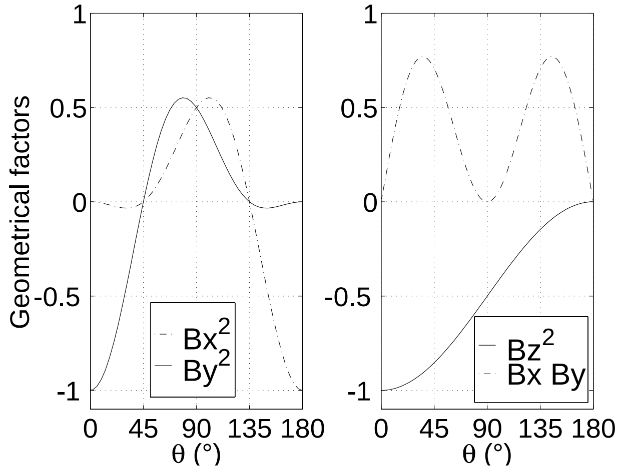

Fig. 2. Geometrical factors appearing in the coefficient C of (15). Left figure (−·): factor for $\mathcal{T}\left(B_x^2\right)$, (−): $\mathcal{T}\left(B_y^2\right)$. Right figure: (−): $\mathcal{T}\left(B_z^2\right)$, (−·): $\mathcal{T}\left(B_x\, B_y\right)$.

by iteration so that we write at first order

$$\begin{aligned} \mathbf{v} \quad &= \mathbf{v}_i + \mathbf{v}_s, \\ \mathbf{v}_s &= \mathcal{G} * S(\mathbf{B}_0, \mathbf{v}_i). \end{aligned} \tag{13}$$

Up to now, the derivation is valid both for time dependent and static magnetic fields. Consider now the latter case. We use Cartesian coordinates x, y, z associated to unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$. We suppose that the incident wave is $\mathbf{v}_i = v_i e^{i(\mathbf{k}_0\cdot\mathbf{r}-\nu_0\, t)}\mathbf{u}_0$ with $\mathbf{u}_0 = \mathbf{i}$ and calculate the scattered wave in the $\mathbf{u}$ direction, as defined in Figure 1. We define $\cos\theta = \mathbf{u}\cdot\mathbf{u}_0$ and note B_x, B_y, B_z the components of $\mathbf{B}_0$. After integration by parts of the terms containing spatial derivatives of the magnetic field, the following expression for the scattered velocity is obtained

$$\mathbf{v}_s = v_i \frac{\nu_0^2 \mathcal{C}}{\rho_0\, \mu_0\, c^4} \frac{e^{i\nu_0(\|\mathbf{r}\|/c-t)}}{4\,\pi\,\|\mathbf{r}\|}\mathbf{u}. \tag{14}$$

The coefficient $\mathcal{C}$ is defined by the following expressions

$$\begin{aligned} \mathcal{C} &= \mathcal{T}\left((\mathbf{B_0}\cdot\mathbf{u}_0)^2\right) - \frac{1+\mathbf{u}_0\cdot\mathbf{u}}{2}\mathcal{T}\left(\mathbf{B_0}^2\right) \\ &\quad - 2\mathbf{u_0}\cdot\mathbf{u}\mathcal{T}\left((\mathbf{B_0}\cdot\mathbf{u}_0)(\mathbf{B_0}\cdot\mathbf{u})\right) \\ &\quad + (1+\mathbf{u_0}\cdot\mathbf{u})\mathcal{T}\left((\mathbf{B_0}\cdot\mathbf{u})^2\right) \\ &= \frac{1-\cos\theta - 2\,\cos^2\theta + 2\,\cos^3\theta}{2}\mathcal{T}\left(B_x^2\right) \\ &\quad + \frac{1+\cos\theta - 2\,\cos^2\theta - 2\,\cos^3\theta}{2}\mathcal{T}\left(B_y^2\right) \\ &\quad - \frac{1+\cos\theta}{2}\mathcal{T}\left(B_z^2\right) + 2\,\cos^2\theta\sin\theta\mathcal{T}\left(B_x\,B_y\right), \end{aligned} \tag{15}$$

where $\mathcal{T}(f) = \int d\mathbf{r}' e^{-i\mathbf{q}\cdot\mathbf{r}'} f(\mathbf{r}')$ is the spatial Fourier transform, with $\mathbf{q} = \frac{\nu_0}{c}(\mathbf{u}-\mathbf{u}_0)$ the scattering wavevector.

The relative amplitude of the scattered wave at a distance r is $v_s/v_i \approx (c_a/c)^2\, l_b^3/(\lambda^2 r)$ where λ is the wavelength, l_b the magnetic field characteristic size and c_a the Alfven waves velocity (see above). In liquid sodium and for a magnetic field $3\,T$, this ratio is around $10^{-3} - 10^{-2}$. Thus, the scattered wave can be measured experimentally.

In the long wavelength limit, the scattered amplitude behaves like λ^{-2}, and for short wavelengths the Born approximation breaks down. The optical theorem is not verified either because the calculation is limited to the first order in the Born approximation [1].

We have given two expressions for $\mathcal{C}$, a compact one and one that uses the Cartesian component of the magnetic field. We now focus on the latter. The angular dependence of the scattered wave comes from purely geometrical coefficients and from the spatial Fourier transform through the scattering wavevector. The purely geometrical coefficients are plotted in Figure 2. Since these coefficients do not vanish simultaneously, there is no extinction that would be independent of the form of the magnetic field. In transmission, for $\theta = 0$, the scattered wave is directly proportional to $B_y^2 + B_z^2$, thus to the magnetic energy transverse to the incident direction. This can be understood since an acoustic perturbation is not coupled to a magnetic field parallel to its direction of propagation. In backscattering, for $\theta = \pi$, the scattered wave depends only on the B_x^2 term. The operation $\theta \to -\theta$ changes only the sign of the coefficient related to $B_x B_y$. This can be understood if we consider the symmetry by respect to the plane $(\mathbf{i}, \mathbf{k})$ that gives $\theta \to -\theta$ and $(B_x, B_y, B_z) \to (-B_x, B_y, -B_z)$.

If the applied field is time dependent, with typical frequencies large compared to the frequency of the acoustic wave, a similar calculation can be done using (12). Consider an harmonic in time magnetic field with frequency Ω: $\tilde{\mathbf{B}}_0 = \mathbf{B_0}\cos\Omega t$, the scattered wave is the sum of three terms of frequencies: $\nu_0 + 2\Omega$, ν_0 and $\nu_0 - 2\Omega$. Each term is given by (14) replacing ν_0 by its corresponding frequency and multiplying by the coefficient $(\nu_0 + 2\Omega)/(4(\nu_0 + \Omega))$, $\nu_0^2/(2(\nu_0^2 - \Omega^2))$ and $(\nu_0 - 2\Omega)/(4(\nu_0 - \Omega))$ respectively. The shift in frequency is the Doppler effect of the sound-magnetic field interaction. Since the interaction term S is quadratic in the magnetic field, the frequency shifts are $2\,\Omega$, 0 and $-2\,\Omega$. This effect can be used in experiments

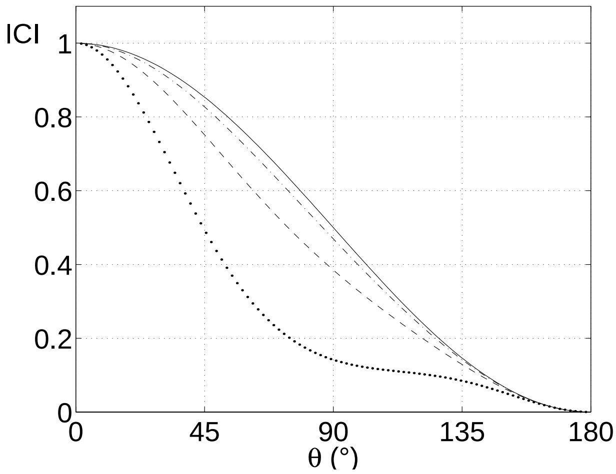

Fig. 3. Absolute value of the coefficient C of (17) for $\pi R^2 B_0^2 = 1$. ($-$): $R/\lambda = .1$, ($-\cdot$): $R/\lambda = .5$, ($--$): $R/\lambda = 1$, ($\cdot$): $R/\lambda = 2$.

to discriminate the scattered wave from the reflection of the incident wave on the boundaries of the experiment.

This calculation must be modified to tackle situations that are invariant with respect to translations parallel to an axis, say $\mathbf{k}$. We then study the scattering in the $(\mathbf{i}, \mathbf{j})$ plane and the spatial integrations are performed along x and y only. The other change traces back to the asymptotic expansion of the Green function for such two dimensional problems (2-D) [1]. In that case the scattered wave is

$$\mathbf{v}_s = v_i \frac{2\sqrt{2\pi} e^{-i\pi/4}}{\rho_0\, \mu_0\, c^2} \mathcal{C} \left(\frac{\nu_0}{c}\right)^{3/2} \frac{e^{i\nu_0(\|\mathbf{r}\|/c - t)}}{\sqrt{\|\mathbf{r}\|}} \mathbf{u}, \qquad (16)$$

where $\mathcal{C}$ is given by (15) integrated over x and y only.

A simple example is a magnetic field tube of radius R, $\mathbf{B}_0 = H(R - r)\, B_0 \mathbf{k}$ where R is the radius of the tube and H is the Heaviside function. In this case the scattered wave calculated with (16) is proportional to

$$\mathcal{C} = -\pi R^2 B_0^2 \frac{1 + \cos\theta}{2} \frac{2J(1, qR)}{qR}, \qquad (17)$$

where $q = \sqrt{2 - 2\cos^2\theta} R/\lambda$ is the norm of the scattering wavevector, $\lambda = c/\nu$ is the wavelength and $J(1, z)$ is the Bessel function of order 1. In Figure 3, we plot the absolute value of $\mathcal{C}$ as a function of θ for fixed value of $\pi R^2 B_0^2$ and for different values of R/λ. As discussed before, there is no backscattering in that case. Most of the scattering is in the incident direction. For wavelengths bigger than the size of the magnetic tube R, the scattering pattern tends to its asymptotic value proportional to $1+\cos\theta$. Note that for $\lambda \ll R$ the Born approximation does not apply.

We now want to be more explicit about what is meant by localized magnetic field. In order to achieve finite values of $\mathcal{C}$, the magnetic field must decrease faster than $r^{-3/2}$ in 3-D and r^{-1} in 2-D, otherwise the first order in the Born approximation breaks down. Thus, the result can not be applied to the field generated by a straight current filament. It is possible that in that situation Aharonov-Bohn effects occur as in the case of the surface wave scattering by a vortex with non-vanishing circulation [1].

We have calculated the acoustic wave scattered by a localized magnetic field. The amplitude of this wave is a function of the magnetic field and measuring it is a non-intrusive probe of the magnetic field. If the field oscillates at frequency Ω, part of the scattered wave is shifted by Doppler effect from frequency ν_0 to $\nu_0 + 2\Omega$ and $\nu_0 - 2\Omega$. For 2-D problems, a similar expression for the scattered wave is calculated and some simple configurations for the magnetic field are discussed. Using equation (14) in 3-D or (16) in 2-D, the wave scattered by a localized magnetic field can be numerically computed.

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